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On Best Simultaneous Approximation

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The problem is considered of best approximation of finite number of functions simultaneously. For a very general class of norms, characterization results are derived. The main part of the paper is concerned with proving uniqueness and strong uniqueness theorems. For a particular subclass, which includes the important special case of the Chebyshev norm, a characterization is given of a uniqueness element.

1. INTRODUCTION

Let X be a compact Hausdorff space and Y a normed linear space with norm $\|\cdot\|_Y$. Let $C(X, Y)$ denote the set of all continuous functions from X to Y, and let $\|\cdot\|_A$ be a norm on $C(X, Y)$. Let U be defined by

$$U=\{\mathbf{a}\in\mathbb{R}^l, \|\mathbf{a}\|_B\leqslant 1\},$$

where $\|\cdot\|_B$ is a given norm on $\mathbb{R}^l$. Define a norm on l-tuples of elements of $C(X, Y)$ as follows: for any $F=(\phi_1, ..., \phi_l)\in C(X, Y)^l$ define

$$\|F\|=\max_{\mathbf{a}\in U}\left\|\sum_{i=1}^{l} a_i\phi_i\right\|_A, \tag{1.1}$$

where $\mathbf{a}=(a_1, ..., a_l)$.

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Now suppose that functions $\phi_1, ..., \phi_l$ in $C(X, Y)$ are given. Then the problem is considered here of approximating these functions simultaneously by functions in S, a subspace of $C(X, Y)$, in the sense of the minimization of the norm (1.1). In other words, we want to find an l-tuple $f = (\phi, ..., \phi)$, where $\phi \in S$, to minimize

$$\|F - f\|. \tag{1.2}$$

If such a function f^* exists, it is called *a best simultaneous approximation to* $F = (\phi_1, ..., \phi_l)$. Problems of simultaneous approximation can be viewed as special cases of vector-valued approximation, and some recent work in this area is due to Pinkus [6], who points out that many questions remain unresolved. He is concerned with the question of when a finite dimensional subspace is a unicity space, for some different norms from those considered here. We are also primarily interested in uniqueness questions. Characterization results for linear problems were recently given in [9] based on the derivation of an expression for the directional derivative, and these generalized earlier work of [8]. We being by showing how these results can be obtained in a simpler and more direct manner, which permits their extension to some nonlinear problems. The rest of the paper is concerned with uniqueness and strong uniqueness of best approximations.

In what follows, finite sequences of identical elements identified by $(\phi, ..., \phi)$ will be assumed to be l-tuples.

2. CHARACTERIZATION OF BEST APPROXIMATIONS

Let $C^*(X, Y)$ denote the dual space of $C(X, Y)$, and let W denote the dual unit ball. For $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$ define

$$g_F(\mathbf{a}, w) = \sum_{i=1}^{l} a_i \langle w, \phi_i \rangle, \qquad \text{for all} \quad (\mathbf{a}, w) \in U \times W,$$

where the inner product notation links elements of $C(X, Y)$ and its dual. Note that $U \times W$ is endowed with the product topology, while W is endowed with the weak * topology. Since for any $(\mathbf{a}^0, w^0) \in U \times W$,

$$\begin{aligned} |g_F(\mathbf{a}, w) - g_F(\mathbf{a}^0, w^0)| &= \left| \sum_{i=1}^{l} a_i \langle w, \phi_i \rangle - \sum_{i=1}^{l} a_i^0 \langle w^0, \phi_i \rangle \right| \\ &\leqslant \sum_{i=1}^{l} |a_i - a_i^0| \, |\langle w, \phi_i \rangle| + \sum_{i=1}^{l} |a_i^0| \, |\langle w - w^0, \phi_i \rangle|, \end{aligned}$$

it follows that $g_F(\cdot,\ \cdot) \in C(U \times W)$ (the space of continuous functions defined on $U \times W$). Note that for any $\phi \in S$, $f = (\phi, ..., \phi)$,

$$g_f(\mathbf{a}, w) = \left(\sum_{i=1}^{l} a_i \right) \langle w, \phi \rangle \in C(U \times W).$$

Further for any such f,

$$\begin{aligned} \|F - f\| &= \max_{\mathbf{a} \in U} \left\| \sum_{i=1}^{l} a_i(\phi_i - \phi) \right\|_A \\ &= \max_{\mathbf{a} \in U} \max_{w \in W} \left| \sum_{i=1}^{l} a_i \langle w, \phi_i - \phi \rangle \right| \\ &= \|g_F(\cdot,\ \cdot) - g_f(\cdot,\ \cdot)\|_C, \end{aligned}$$

where $\|\cdot\|_C$ denotes the uniform norm on $C(U \times W)$. Now define

$$S_g = \{ g_f : f = (\phi, ..., \phi), \phi \in S \}.$$

It follows that $f^* = (\phi^*, ..., \phi^*)$, $\phi^* \in S$ is a best simultaneous approximation to $F = (\phi_1, ..., \phi_l)$ if and only if $g_{f^*} \in S_g$ is a best approximation to g_F in the uniform norm of $C(U \times W)$. Let $P_S(F)$ denote the set of all best simultaneous approximations $f = (\phi, ..., \phi)$, where $\phi \in S$, to F. In addition, let

$$\begin{aligned} d(F, S) &= \inf \{ \|F - f\| : f = (\phi, ..., \phi), \phi \in S \}, \\ d(F, C) &= d(F, C(X, Y)). \end{aligned}$$

DEFINITION 1. A set S is a *sunset* for simultaneous approximation if for any $F = (\phi_1, ..., \phi_l)$, and $f^* = (\phi^*, ..., \phi^*)$, $\phi^* \in S$, $f^* \in P_S(F)$ implies that $f^* \in P_S(F_\alpha)$ for $F_\alpha = f^* + \alpha(F - f^*)$, and $\alpha \geqslant 0$.

Descriptions of sunsets (or strict suns), and solar properties, are given in [3]. Linear sets are examples of suns, as are convex sets, but also some non-convex sets, for example rational functions.

THEOREM 1. *Let $S \subset C(X, Y)$ be a sunset of simultaneous approximation. Then $f^* \in P_S(F)$ if and only if for any $f = (\phi, ..., \phi)$, $\phi \in S$, there exists $\mathbf{a} \in \operatorname{ext}(U)$, $w \in \operatorname{ext}(W)$ such that*

$$\begin{aligned} g_{F-f^*}(\mathbf{a}, w) &= \|F - f^*\|, \\ g_{f^*-f}(\mathbf{a}, w) &\geqslant 0, \end{aligned}$$

where "ext" denotes the set of extreme points.

Proof. If S is a sunset for simultaneous approximation, then S_g is a strict sun for uniform approximation in $C(U \times W)$. The result then follows from the generalized Kolmogorov criterion characterizing a best approximation in $C(U \times W)$ with the uniform norm (see, for example [3, Theorem I.2.4]), using the Krein–Milman Theorem. ∎

A special case of (1.1) is given by

$$\|\phi\|_A = \max_{t \in X} \|\phi(t)\|_Y, \tag{2.1}$$

for any $\phi \in C(X, Y)$. This includes the important case of the Chebyshev norm on l-tuples. Let $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$. Then a Chebyshev norm may be defined by

$$\|F\| = \max_{1 \leqslant i \leqslant l} \max_{t \in X} \|\phi_i(t)\|_Y, \tag{2.2}$$

which is the special case of (1.1) when $\|\cdot\|_A$ is given by (2.1) and $\|\cdot\|_B$ is the l_1 norm [see 8, 9]. Define

$$H(F, f) = \left\{ t \in X \colon \max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i(\phi_i(t) - \phi(t)) \right\|_Y = \|F - f\| \right\}. \tag{2.3}$$

Let Z denote the unit ball in Y^*, the dual space of Y, and let $\langle \cdot, \cdot \rangle_Y$ denote the inner product linking Y and Y^*. Then using the form of points in ext(W) in this case, we have the following corollary of Theorem 1.

COROLLARY 1. *Let $\|\cdot\|_A$ be given by (2.1), and let $S \subset C(X, Y)$ be a sunset for simultaneous approximation. Then $f^* \in P_S(F)$ if and only if for any $f = (\phi, ..., \phi)$, $\phi \in S$, there exists $\mathbf{a} \in \operatorname{ext}(U)$, $t \in H(F, f)$, $v(t) \in \operatorname{ext}(Z)$ such that*

$$\sum_{i=1}^{l} a_i \langle v(t), \phi_i(t) - \phi^*(t) \rangle_Y = \|F - f^*\|,$$

$$\left(\sum_{i=1}^{l} a_i \right) \langle v(t), \phi^*(t) - \phi(t) \rangle_Y \geqslant 0.$$

Returning to the general problem, standard linear theory (for example [7]) gives the following result.

THEOREM 2. *Let S be an n-dimensional subspace of $C(X, Y)$. Then $f^* \in P_S(F)$ if and only if there exists $\mathbf{a}^j \in \operatorname{ext}(U)$, $w^j \in \operatorname{ext}(W)$, $\alpha_j > 0$, $j = 1, ..., r$ with $\sum_{j=1}^{r} \alpha_j = 1$ and $1 \leqslant r \leqslant n+1$ such that*

$$g_{F-f^*}(\mathbf{a}^j, w^j) = \|F - f^*\|, \qquad j = 1, ..., r,$$

$$\sum_{j=1}^{r} \alpha_j g_f(\mathbf{a}^j, w^j) = 0 \qquad \text{for all} \quad f \in S.$$

This is just the result given as Theorem 1 in [9].

3. UNIQUENESS OF BEST APPROXIMATIONS

For the general case uniqueness is a consequence of strict convexity of the norm $\|\cdot\|_A$. This is established next. It is convenient to extract the following result as a preliminary lemma.

LEMMA 1. *Let* $S \subset C(X, Y)$, *and let* $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$. *Let* $f^* = (\phi^*, ..., \phi^*) \in P_S(F)$, *with* $\mathbf{a}^* \in U$ *such that*

$$\|F - f^*\| = \left\| \sum_{i=1}^{l} a_i^*(\phi_i - \phi^*) \right\|_A. \tag{3.1}$$

Then if $d(F, C) < d(F, S)$, $\sum_{i=1}^{l} a_i^* \neq 0$.

Proof. Assume that $d(F, C) < d(F, S)$ and also that $\mathbf{a}^*$ satisfying (3.1) is such that $\sum_{i=1}^{l} a_i^* = 0$. Then there exists $\phi \in C(X, Y)$ such that

$$\max_{\mathbf{a} \in U} \left\| \sum_{i=1}^{l} a_i(\phi_i - \phi) \right\|_A < \left\| \sum_{i=1}^{l} a_i^*(\phi_i - \phi^*) \right\|_A$$

$$= \left\| \sum_{i=1}^{l} a_i^*(\phi_i - \phi) \right\|_A.$$

This is a contradiction which proves the result. ∎

THEOREM 3. *Let* $\|\cdot\|_A$ *be strictly convex, and let* $S \subset C(X, Y)$ *be a sunset for simultaneous approximation. Then for any* $F = (\phi_1, ..., \phi_l)$, $d(F, C) = d(F, S)$ *or* $P_S(F)$ *contains at most one element.*

Proof. Let $F = (\phi_1, ..., \phi_l)$ be such that $d(F, C) < d(F, S)$. Suppose that $f^* = (\phi^*, ..., \phi^*) \in P_S(F)$, $\bar{f} = (\bar{\phi}, ..., \bar{\phi}) \in P_S(F)$ with $\phi^* \neq \bar{\phi}$. Let $\phi_i^0 = 2\phi_i - \bar{\phi}$, $i = 1, ..., l$, $F^0 = (\phi_1^0, ..., \phi_l^0)$. Then it follows from the definition of a sunset that $\bar{f} \in P_S(F^0)$. Also

$$\begin{aligned}\|F^0 - f^*\| &= \|2F - \bar{f} - f^*\| \\ &\leqslant \|F - \bar{f}\| + \|F - f^*\| \\ &= 2\,\|F - \bar{f}\| \\ &= \|F^0 - \bar{f}\|. \end{aligned} \tag{3.2}$$

Therefore $f^* \in P_S(F^0)$.

Further

$$\begin{aligned}\|2F - \bar{f} - f^*\| &= \max_{\mathbf{a} \in U} \left\| \sum_{i=1}^{l} a_i (2\phi_i - \bar{\phi} - \phi^*) \right\|_A \\ &\leqslant \max_{\mathbf{a} \in U} \left[\left\| \sum_{i=1}^{l} a_i (\phi_i - \bar{\phi}) \right\|_A + \left\| \sum_{i=1}^{l} a_i (\phi_i - \phi^*) \right\|_A \right] \\ &\leqslant \|F^0 - \bar{f}\|, \qquad \text{using (3.2).} \end{aligned} \tag{3.3}$$

Thus equality holds to (3.3). Let $\mathbf{a} \in U$ be chosen so that

$$\left\| \sum_{i=1}^{l} (2\phi_i - \bar{\phi} - \phi^*) \right\|_A = \|2F - \bar{f} - f^*\| = 2d(F, C).$$

Thus

$$\begin{aligned}&\left\| \sum_{i=1}^{l} a_i (\phi_i - \bar{\phi}) + \sum_{i=1}^{l} a_i (\phi_i - \phi^*) \right\|_A \\ &\qquad = \left\| \sum_{i=1}^{l} a_i (\phi_i - \bar{\phi}) \right\|_A + \left\| \sum_{i=1}^{l} a_i (\phi_i - \phi^*) \right\|_A, \end{aligned}$$

which, using the assumption of strict convexity, implies that

$$\sum_{i=1}^{l} a_i (\phi_i - \bar{\phi}) = \sum_{i=1}^{l} a_i (\phi_i - \phi^*)$$

or

$$\left(\sum_{i=1}^{l} a_i \right) (\phi^* - \bar{\phi}) = 0. \tag{3.4}$$

Since by assumption $d(F, C) < d(F, S)$, it follows from Lemma 1 that $(\sum_{i=1}^{l} a_i) \neq 0$, and so $\phi^* = \bar{\phi}$ and the result is established. ∎

For the special case covered by (2.1), it is possible to give a precise characterization of a uniqueness element, which is defined as follows.

Definition 2. Let S be a sunset of $C(X, Y)$, and $f^* = (\phi^*, ..., \phi^*)$, with $\phi^* \in S$. Then ϕ^* is called a uniqueness element of S if for any $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$ with $d(F, C) < d(F, S)$, $f^* \in P_S(F)$, then f^* is a unique best approximation to F from S.

Theorem 4. *Let Y be strictly convex, let $\|\cdot\|_A$ be given by* (2.1), *and let S be a sunset for simultaneous approximation. Let $f^* = (\phi^*, ..., \phi^*) \in S^l$. Then ϕ^* is a uniqueness element of S if and only if for any $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$, with $f^* \in P_S(F)$ and $d(F, C) < d(F, S)$, ϕ^* is uniquely determined by the set of values in $H(F, f^*)$ (that is, if $\phi \in S$ and $\phi(t) = \phi^*(t)$ for all $t \in H(F, f^*)$, then $\phi = \phi^*$).*

Proof. Let ϕ^* be a uniqueness element of S. Suppose that for some $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$, with $f^* \in P_S(F)$ and $d(F, C) < d(F, S)$, there exists $\bar{f} = (\bar{\phi}, ..., \bar{\phi})$, $\bar{\phi} \in S$, $\bar{\phi} \neq \phi^*$, such that

$$\bar{\phi}(t) = \phi^*(t), \qquad \text{for all} \quad t \in H(F, f^*).$$

Define for all $t \in X$

$$\phi_i^0(t) = \phi^*(t) + \left[\|\phi^* - \bar{\phi}\|_A - \max_{\|\mathbf{a}\|_B = 1} \left(\sum_{j=1}^{l} a_j \right) \|\phi^*(t) - \bar{\phi}(t)\|_Y \right] \times \frac{\phi_i(t) - \phi^*(t)}{\|F - f^*\|}, \tag{3.5}$$

and let

$$F^0 = (\phi_1^0, ..., \phi_l^0).$$

It is easy to verify directly that

$$\|F^0 - f^*\| \leqslant \|\phi^* - \bar{\phi}\|_A. \tag{3.6}$$

Since for any $t \in H(F, f^*)$,

$$\max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i(\phi_i^0(t) - \phi^*(t)) \right\|_Y = \|\phi^* - \bar{\phi}\|_A,$$

then

$$\|F^0 - f^*\| \geqslant \|\phi^* - \bar{\phi}\|_A. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\|F^0 - f^*\| = \|\phi^* - \bar{\phi}\|_A. \tag{3.8}$$

From Theorem 1, because $f^* \in P_S(F)$, for any $\phi \in S$, there exists $\mathbf{a} \in \text{ext}(U)$, $w \in \text{ext}(W)$, such that

$$\left\langle w, \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\rangle = \|F - f^*\|, \qquad \left(\sum_{i=1}^{l} a_i\right) \langle w, \phi^* - \phi \rangle \geqslant 0, \tag{3.9}$$

and so using Corollary 1, for some $t \in H(F, f^*)$, $v(t) \in \text{ext}(Z)$,

$$\sum_{i=1}^{l} a_i \langle v(t), \phi_i(t) - \phi^*(t) \rangle_Y = \|F - f^*\|.$$

From (3.5),

$$\sum_{i=1}^{l} a_i \langle v(t), \phi_i^0(t) - \phi^*(t) \rangle_Y = \|\phi^* - \bar{\phi}\|_A$$
$$= \|F^0 - f^*\|, \qquad \text{using (3.8)}.$$

Thus $t \in H(F^0, f^*)$, and we must have

$$\left\langle w, \sum_{i=1}^{l} a_i(\phi_i^0 - \phi^*) \right\rangle = \|F^0 - f^*\|. \tag{3.10}$$

Equations (3.9) and (3.10) show, using Theorem 1, that $f^* \in P_S(F^0)$.

Now for any $t \in X$,

$$\max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i(\phi_i^0(t) - \bar{\phi}(t)) \right\|_Y$$
$$\leqslant \max_{\|\mathbf{a}\|_B = 1} \left\| \left(\sum_{i=1}^{l} a_i\right)(\phi^*(t) - \bar{\phi}(t)) \right\|_Y$$
$$+ \max_{\|\mathbf{a}\|_B = 1} \left\| \left[\|\phi^* - \bar{\phi}\|_A - \max_{\|\mathbf{a}\|_B = 1} \left\| \left(\sum_{i=1}^{l} a_i\right)(\phi^*(t) - \bar{\phi}(t)) \right\|_Y \right] \right.$$
$$\left. \times \frac{1}{\|F - f^*\|} \sum_{i=1}^{l} a_i(\phi_i(t) - \phi^*(t)) \right\|_Y$$
$$\leqslant \|\phi^* - \bar{\phi}\|_A.$$

Thus

$$\|F^0 - \bar{f}\| \leqslant \|F^0 - f^*\|, \qquad \text{using (3.8)},$$

so that $\bar{f} \in P_S(F^0)$, a contradiction. This proves necessity.

Now suppose that for some $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$ with $d(F, C) < d(F, S)$ and $f^* \in P_S(F)$, $f = (\phi^*, ..., \phi^*)$, there exists another $\bar{f} \in P_S(F)$, $\bar{f} = (\bar{\phi}, ..., \bar{\phi})$. Let

$$\phi_i^0 = 2\phi_i - \bar{\phi}, \qquad i = 1, ..., l,$$

and let

$$F^0 = (\phi_1^0, ..., \phi_l^0).$$

It follows from the definition of a sunset that $\bar{f} \in P_S(F^0)$. Now

$$\begin{aligned} \|F^0 - f^*\| &= \max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i (2\phi_i - \bar{\phi} - \phi^*) \right\|_A \\ &\leqslant \max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i (\phi_i - \bar{\phi}) \right\|_A + \max_{\|\mathbf{a}\|_B = 1} \left\| \sum_{i=1}^{l} a_i (\phi_i - \phi^*) \right\|_A \\ &= 2\, \|F - \bar{f}\| \\ &= \|F^0 - \bar{f}\|. \end{aligned}$$

Thus $f^* \in P_S(F^0)$. Now let $\mathbf{a} \in \text{ext}(U)$, $w \in \text{ext}(W)$ such that

$$\left\langle w, \sum_{i=1}^{l} a_i (2\phi_i - \bar{\phi} - \phi^*) \right\rangle = \|F^0 - f^*\|,$$

which is possible using Theorem 1. Thus for any $t \in H(F^0, f^*)$,

$$\begin{aligned} \|F^0 - f^*\| &= \left\| \sum_{i=1}^{l} a_i (2\phi_i - \bar{\phi} - \phi^*)(t) \right\|_Y \\ &\leqslant \left\| \sum_{i=1}^{l} a_i (\phi_i - \bar{\phi})(t) \right\|_Y + \left\| \sum_{i=1}^{l} a_i (\phi_i - \phi^*)(t) \right\|_Y \\ &\leqslant \|F - \bar{f}\| + \|F - f^*\| \\ &= \|F^0 - f^*\|. \end{aligned}$$

It follows that

$$\left\| \sum_{i=1}^{l} a_i(\phi_i - \bar{\phi})(t) + \sum_{i=1}^{l} a_i(\phi_i - \phi^*)(t) \right\|_Y$$
$$= \left\| \sum_{i=1}^{l} a_i(\phi_i - \bar{\phi})(t) \right\|_Y + \left\| \sum_{i=1}^{l} a_i(\phi_i - \phi^*)(t) \right\|_Y.$$

Therefore using the strict convexity of Y,

$$\sum_{i=1}^{l} a_i(\phi_i - \bar{\phi})(t) = \sum_{i=1}^{l} a_i(\phi_i - \phi^*)(t),$$

or equivalently

$$\left(\sum_{i=1}^{l} a_i \right) (\phi^*(t) - \bar{\phi}(t)) = 0, \qquad \text{for all} \quad t \in H(F^0, f^*).$$

Since $d(F, C) < d(F, S)$, by Lemma 1 we must have $\sum_{i=1}^{l} a_i \neq 0$, and so

$$\phi^*(t) = \bar{\phi}(t) \qquad \text{for all} \quad t \in H(F^0, f^*).$$

This proves the sufficiency of the stated conditions. ∎

4. STRONG UNIQUENESS

It is possible to establish strong uniqueness for the general problem under a condition which generalizes the Chebyshev set condition for linear best approximation in the uniform norm. The result hinges on the derivation of the analogue of the strong Kolomogorov condition for finite dimensional spaces (see, for example, Wulbert [10] or Nurnberger [5]).

DEFINITION 3 [1]. An n-dimensional subspace S of $C(X, Y)$ is called an *interpolating subspace* if no nontrivial linear combination of n linearly independent extreme points of W annihilates S.

THEOREM 5. *Let S be an interpolating subspace of $C(X, Y)$, and let $d(F, C) < d(F, S)$. Then $f^* = (\phi^*, ..., \phi^*) \in P_S(F)$ is a strongly unique best simultaneous approximation to F, that is, there exists $\gamma > 0$ such that*

$$\|F - f\| \geqslant \|F - f^*\| + \gamma \, \|\phi - \phi^*\|_A \qquad \textit{for all} \quad f = (\phi, ..., \phi), \qquad \phi \in S.$$

Proof. We can use Theorem 2. Since $f^* \in P_S(F)$, it follows that there exist $\mathbf{a}^j \in \text{ext}(U)$, $w^j \in \text{ext}(W)$, $\alpha_j > 0$, $j = 1, ..., r$ with $\sum_{j=1}^{r} \alpha_j = 1$, and $1 \leqslant r \leqslant n+1$ such that

$$\sum_{i=1}^{l} a_i^j \langle w^j, \phi_i - \phi^* \rangle = \|F - f^*\|, \qquad j = 1, ..., r, \tag{4.1}$$

$$\sum_{j=1}^{r} \alpha_j \left(\sum_{i=1}^{l} a_i^j \right) \langle w^j, \phi^* - \phi \rangle = 0 \qquad \text{for all} \quad \phi \in S. \tag{4.2}$$

With no loss of generality, we can assume that the set $\{w^j, j = 1, ..., r\}$ is linearly independent. Because $d(F, C) < d(F, S)$, it follows from Lemma 1 that $\beta_j = \alpha_j (\sum_{i=1}^{l} a_i^j) \neq 0$, $j = 1, ..., n$. Assume that $r < n+1$. Then we can take an element $\phi_0 \in S$ such that $\phi_0 \neq \phi^*$, and $\langle w^j, \phi^* - \phi_0 \rangle = \beta_j, j = 1, ..., r$, using the fact that S is an interpolating subspace. This means that

$$\sum_{j=1}^{r} \beta_j \langle w^j, \phi^* - \phi_0 \rangle = \sum_{j=1}^{r} \beta_j^2 > 0,$$

which is a contradiction. It follows that $r = n+1$. Now for any $\phi \in S$, any $(\mathbf{a}^j, w^j) \in \text{ext } U \times \text{ext } W$ satisfying (4.1) and (4.2) assume that we have

$$\left(\sum_{i=1}^{l} a_i^j \right) \langle w^j, \phi^* - \phi \rangle \leqslant 0, \qquad j = 1, ..., r.$$

Then it follows from (4.2) that

$$0 = \sum_{j=1}^{r} \beta_j \langle w^j, \phi^* - \phi \rangle \leqslant 0,$$

which in turn implies that

$$\langle w^j, \phi^* - \phi \rangle = 0, \qquad j = 1, ..., r,$$

or $\phi^* = \phi$, since $r = n+1$. Thus there exists $(\mathbf{a}, w) \in \text{ext } U \times \text{ext } W$ with

$$\sum_{i=1}^{l} a_i \langle w, \phi_i - \phi^* \rangle = \|F - f^*\|,$$

$$\left(\sum_{i=1}^{l} a_i \right) \langle w, \phi^* - \phi \rangle > 0.$$

Now define

$$M(F, f^*) = \left\{ (\mathbf{a}, w) \in \text{ext}(U) \times \text{ext}(W) \colon \sum_{i=1}^{l} a_i \langle w, \phi_i - \phi^* \rangle = \|F - f^*\| \right\}.$$

Then we have

$$\max_{(\mathbf{a}, w) \in M(F, f^*)} \left(\sum_{i=1}^{l} a_i \right) \langle w, \phi^* - \phi \rangle > 0, \qquad \text{for all} \quad f \in S.$$

Let

$$\gamma = \inf_{\phi \in S \setminus \{\phi^*\}} \max_{(\mathbf{a}, w) \in M(F, f^*)} \left\{ \left(\sum_{i=1}^{l} a_i \right) \left\langle w, \frac{\phi^* - \phi}{\|f^* - f\|} \right\rangle \right\}.$$

Then $\gamma > 0$, since S is finite dimensional. Further for any $\phi \in S$,

$$\sum_{i=1}^{l} a_i \langle w, \phi_i - \phi \rangle = \sum_{i=1}^{l} a_i \langle w, \phi_i - \phi^* \rangle + \left(\sum_{i=1}^{l} a_i \right) \langle w, \phi^* - \phi \rangle.$$

For any $(\mathbf{a}, w) \in M(F, f^*)$, it follows that

$$\begin{aligned} \|F - f\| - \|F - f^*\| &\geqslant \max_{(\mathbf{a}, w) \in M(F, f^*)} \left(\sum_{i=1}^{l} a_i \right) \langle w, \phi^* - \phi \rangle \\ &\geqslant \gamma \, \|\phi^* - \phi\|_A. \end{aligned}$$

This implies that

$$\|F - f\| \geqslant \|F - f^*\| + \gamma \, \|\phi - \phi^*\|_A \qquad \text{for all} \quad f \in S,$$

and the proof is complete. ∎

Definition 4. For any normed linear space $\{E, \|\cdot\|\}$, the *modulus of convexity* is defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \tfrac{1}{2} \|x + y\| : x, y \in E, \|x - y\| = \varepsilon, \|x\| = \|y\| = 1 \right\},$$

for $0 < \varepsilon \leqslant 2$.

Definition 5 [2, 4, 11]. E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ for any $0 < \varepsilon \leqslant 2$. A uniformly convex space E is *p-uniformly convex* (or has *modulus of convexity of power type p*) if for some $c > 0$, $\delta_E(\varepsilon) \geqslant c\varepsilon^p$.

Examples of uniformly convex spaces are Hilbert spaces and the L_p spaces, $1 < p < \infty$. In fact, L_p spaces are 2-uniformly convex if $1 < p \leqslant 2$, and p-uniformly convex if $p > 2$.

Remark. Let

$$d_p = \inf\left\{\frac{\frac{1}{2}\|x\|^p + \frac{1}{2}\|y\|^p - \|\frac{1}{2}(x+y)\|^p}{\|x-y\|^p}, x, y \in E, \|x-y\| > 0\right\}. \quad (4.3)$$

Then it follows from [2] or [11] that $d_p > 0$ if and only if E is p-uniformly convex.

THEOREM 6. *Let $C(X, Y)$ be p-uniformly convex, let S be a convex subset of $C(X, Y)$, and let $F = (\phi_1, ..., \phi_l) \in C(X, Y)^l$, with $d(F, C) < d(F, S)$. Then $f^* = (\phi^*, ..., \phi^*) \in P_C(F)$ is a strongly unique best simultaneous approximation of order p to F, that is, there exists $\gamma_p > 0$ such that*

$$\|F - f\|^p \geqslant \|F - f^*\|^p + \gamma_p \|\phi - \phi^*\|_A^p, \qquad \text{for all} \quad f = (\phi, ..., \phi), \qquad \phi \in S.$$

Proof. Let the stated conditions hold, and let

$$\gamma_p(f) = \frac{\|F - f\|^p - \|F - f^*\|^p}{\|\phi - \phi^*\|_A^p}.$$

Then it is sufficient to prove that

$$\inf\{\gamma_p(f) : f = (\phi, ..., \phi), \phi \in S, \phi \neq \phi^*\} > 0.$$

Without loss of generality, suppose that $C(X, Y)$ is complete. Suppose also that there exists a sequence $\{f_n\}$, with $f_n = (\psi_n, ..., \psi_n)$, $\psi_n \in S$, $\psi_n \neq \phi^*$ such that

$$\gamma_p(f_n) \to 0, \qquad \text{as} \quad n \to \infty. \quad (4.4)$$

We will show that this leads to a contradiction. Now

$$\begin{aligned}\gamma_p(f_n) &= \frac{\|F - f_n\|^p - \|F - f^*\|^p}{\|\psi_n - \phi^*\|_A^p} \\ &\geqslant \left[\frac{|\,\|f_n - f^*\| - \|F - f^*\|\,|}{\|\psi_n - \phi^*\|_A}\right]^p - \left[\frac{\|F - f^*\|}{\|\psi_n - \phi^*\|_A}\right]^p \\ &\geqslant \left[\max_{\mathbf{a} \in U} \sum_{i=1}^{l} a_i - \frac{\|F - f^*\|}{\|\psi_n - \phi^*\|_A}\right]^p - \left[\frac{\|F - f^*\|}{\|\psi_n - \phi^*\|_A}\right]^p.\end{aligned}$$

Thus if $\{\psi_n\}$ is unbounded, it follows that

$$\|\psi_n - \phi^*\|_A \to \infty, \qquad \text{as} \quad n \to \infty.$$

This in turn implies that

$$\lim_{n \to \infty} \gamma_p(f_n) \geqslant \max_{\mathbf{a} \in U} \sum_{i=1}^{l} a_i > 0,$$

which contradicts (4.4). Thus $\{\psi_n\}$ is bounded, and so

$$\|F - f_n\| \to \|F - f^*\|, \qquad \text{as} \quad n \to \infty,$$

by definition of $\gamma_p(f)$. Thus there exists $\bar{f} = (\bar{\phi}, ..., \bar{\phi})$, $\bar{\phi} \in \text{clo}(S)$, where $\text{clo}(S)$ denotes the closure of S, and a subsequence of $\{f_n\}$ (which we do not rename) such that $\psi_n \to \bar{\phi}$ weakly. Note that $C(X, Y)$ is reflexive by the assumption of p-uniform convexity [4].

Now let $\mathbf{a} = (a_1, ..., a_l) \in U$, $w \in W$ such that

$$\sum_{i=1}^{l} a_i \langle w, \phi_i - \phi^* \rangle = \|F - f^*\|,$$

$$\sum_{i=1}^{l} a_i \langle w, \phi^* - \phi \rangle \geqslant 0, \qquad \text{for all} \quad \phi \in S, \tag{4.5}$$

using Theorem 1. Since $d(F, C) < d(F, S)$, by Lemma 1 $\sum_{i=1}^{l} a_i \neq 0$. Then

$$\begin{aligned}
&\lim_{n \to \infty} \left\| \sum_{i=1}^{l} a_i(\phi_i - \psi_n) + \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\|_A \\
&\geqslant \lim_{n \to \infty} \left\langle w, \sum_{i=1}^{l} a_i(\phi_i - \psi_n) + \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\rangle \\
&= \lim_{n \to \infty} \left\langle w, \sum_{i=1}^{l} a_i(\phi_i - \psi_n) \right\rangle + \|F - f^*\| \\
&= \left\langle w, \sum_{i=1}^{l} a_i(\phi_i - \bar{\phi}) \right\rangle + \|F - f^*\|, \\
&\geqslant \left\langle w, \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\rangle + \|F - f^*\|, \qquad \text{using (4.5)}, \\
&= 2\,\|F - f^*\|.
\end{aligned}$$

Also

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{l} a_i(\phi_i - \psi_n) \right\|_A \leqslant \lim_{n\to\infty} \|F - f_n\| = \|F - f^*\|,$$

and

$$\left\| \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\|_A = \|F - f^*\|,$$

and so

$$\left\| \sum_{i=1}^{l} a_i(\phi_i - \phi_n) - \sum_{i=1}^{l} a_i(\phi_i - \phi^*) \right\|_A \to 0, \qquad \text{as} \quad n \to \infty,$$

using the p-uniform convexity property of $C(X, Y)$. Since $\sum_{i=1}^{l} a_i \neq 0$, it follows that

$$\|f_n - f^*\| \to 0, \qquad \text{as} \quad n \to \infty.$$

Now let the sequence $\{\mathbf{a}^n = (a_1^n, ..., a_l^n)\} \in U$ be such that

$$\left\| \sum_{i=1}^{l} a_i^n(\phi_i - \tfrac{1}{2}(\psi_n + \phi^*)) \right\|_A = \|F - \tfrac{1}{2}(f_n + f^*)\|. \tag{4.6}$$

Then

$$\begin{aligned} \|F - \tfrac{1}{2}(f_n + f^*)\| &\leqslant \tfrac{1}{2} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \psi_n) \right\|_A + \tfrac{1}{2} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \phi^*) \right\|_A \\ &\leqslant \tfrac{1}{2} \|F - f_n\| + \tfrac{1}{2} \|F - f^*\|. \end{aligned}$$

Also

$$\lim_{n\to\infty} \|F - \tfrac{1}{2}(f_n + f^*)\| = \tfrac{1}{2} \lim_{n\to\infty} (\|F - f_n\| + \|F - f^*\|) = \|F - f^*\|.$$

Thus

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \psi_n) \right\|_A = \|F - f^*\|. \tag{4.7}$$

Since $\sum_{i=1}^{l} a_i^n = 0$ implies that $\frac{1}{2}(f_n + f^*) \in P_C(F)$, then

$$d(F, S) \leqslant \|F - \tfrac{1}{2}(f_n + f^*)\| = d(F, C),$$

which contradicts the assumption that $d(F, C) < d(F, S)$. Thus $\sum_{i=1}^{l} a_i^n \neq 0$ for any $n \geqslant 1$. Let

$$\lambda = \inf_n \left| \sum_{i=1}^{l} a_i^n \right|. \tag{4.8}$$

Assume $\lambda = 0$. Then, going to a subsequence if necessary, we must have $a_i^n \to a_i^0$, $i = 1, ..., l$ with $\sum_{i=1}^{l} a_i^0 = 0$. Since $\|f_n - f^*\| \to 0$,

$$\begin{aligned} \left\| \sum_{i=1}^{l} a_i^0(\phi_i - \phi^*) \right\|_A &= \lim_{n\to\infty} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \psi_n) \right\|_A \\ &= \|F - f^*\|, \end{aligned}$$

using (4.7). This implies that $d(F, C) = d(F, S)$, which is a contradiction. Thus $\lambda > 0$. Now for any $n \geqslant 1$,

$$\begin{aligned} \|F - f^*\|^p &\leqslant \|F - \tfrac{1}{2}(f_n + f^*)\|^p \\ &= \left\| \tfrac{1}{2} \left[\sum_{i=1}^{l} a_i^n(\phi_i - \psi_n) + \sum_{i=1}^{l} a_i^n(\phi_i - \phi^*) \right] \right\|_A^p, \qquad \text{by (4.6)}, \\ &\leqslant \tfrac{1}{2} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \psi_n) \right\|_A + \tfrac{1}{2} \left\| \sum_{i=1}^{l} a_i^n(\phi_i - \phi^*) \right\|_A^p \\ &\quad - d_p \left\| \sum_{i=1}^{l} a_i^n(\psi_n - \phi^*) \right\|_A^p \\ &\leqslant \tfrac{1}{2} \|F - f_n\|^p + \tfrac{1}{2} \|F - f^*\|^p - \lambda^p d_p \|\psi_n - \phi^*\|_A^p, \end{aligned}$$

using the above remark and (4.8). Hence

$$\|F - f^*\|^p \leqslant \|F - f_n\|^2 - 2\lambda^p d_p \|\psi_n - \phi^*\|_A^p,$$

and

$$\gamma_p(f_n) \geqslant 2\lambda^p d_p > 0.$$

This is a contradiction and the theorem is proved. ∎

It is easy to give examples of spaces which satisfy the conditions of this Theorem. For example, let Y be a p-uniformly convex Banach space, and let X be a measure space. Then $C(X, Y)$ is p-uniformly convex with the norm

$$\|\phi\|_A = \left\{ \int_X \|\phi(t)\|_Y^p \right\}^{1/p}, \qquad 1 \leqslant p < \infty.$$

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